

Negative heat capacity for a Klein-Gordon oscillator in non-commutative complex phase space

Slimane Zaim*, Hakim Guelmamene and Yazid Delenda

Département de Physique, Faculté des Sciences de la Matière,
Université Batna 1, Algeria.

Abstract

We obtain exact solutions to the two-dimensional Klein-Gordon oscillator in a non-commutative complex phase space to first order in the non-commutativity parameter. We derive the exact non-commutative energy levels and show that the energy levels split to $2m$ levels. We find that the non-commutativity plays the role of a magnetic field interacting automatically with the spin of a particle induced by the non-commutativity of complex phase space. The effect of the non-commutativity parameter on the thermal properties is discussed. It is found that the dependence of the heat capacity C_V on the non-commutative parameter gives rise to a negative quantity. Phenomenologically, this effectively confirms the presence of the effects of self-gravitation induced by the non-commutativity of complex phase space.

KEYWORDS: non-commutative geometry, solutions of wave equations, statistical physics.

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*Corresponding Author, E-mail: zaim69slimane@yahoo.com

1 Introduction

There are many papers in the literature which are devoted to the study of various aspects of the Klein-Gordon oscillator in non-commutative (NC) space and NC phase space with the usual time coordinate [1 – 3]. However the extension of this study to the case of a two-dimensional (2D) NC *complex* space is limited [4, 5]. This topic is still very interesting since its phenomenological implications are important.

This paper is organized as follows. In Section 2, we discuss the Klein-Gordon oscillator in NC complex space. In Section 3, we study the Klein-Gordon oscillator in NC complex phase space. Then, the thermodynamic properties are studied in section 4. Finally, section 5 is devoted to a discussion.

2 2D Klein-Gordon oscillator in NC complex space

In a complex space, the NC complex coordinate operators $(\hat{z}, \hat{\bar{z}})$ and momentum operators $(\hat{p}_z, \hat{p}_{\bar{z}})$ in 2D space are defined by [4]:

$$\hat{z} = \hat{x} + i\hat{y} = z + i\theta p_{\bar{z}}, \quad \hat{\bar{z}} = \hat{x} - i\hat{y} = \bar{z} - i\theta p_z, \quad (1)$$

$$\hat{p}_z = p_z, \quad \hat{p}_{\bar{z}} = p_{\bar{z}}. \quad (2)$$

These operators satisfy the following commutation relations:

$$\begin{aligned} [\hat{z}, \hat{\bar{z}}] &= 2\theta, & [\hat{z}, \hat{p}_z] &= [\hat{\bar{z}}, \hat{p}_{\bar{z}}] = 0, \\ [\hat{z}, \hat{p}_{\bar{z}}] &= [\hat{\bar{z}}, \hat{p}_z] = 2\hbar, & [\hat{p}_z, \hat{p}_{\bar{z}}] &= 0. \end{aligned} \quad (3)$$

Now, following ref. [4], we review the Klein-Gordon oscillator in NC complex space. The Klein-Gordon oscillator in 2D complex space is defined by the following equation:

$$(2p_{\bar{z}} + im\omega z)(2p_z - im\omega \bar{z})\psi = (E^2 - m^2)\psi, \quad (4)$$

which can be rewritten in commutative space as:

$$(p_x^2 + p_y^2 + m^2\omega^2(x^2 + y^2) + 2m\omega L_z)\psi = (E^2 - m^2 + 2m\omega)\psi, \quad (5)$$

with energy eigenvalues:

$$E^2 = 2m\omega(n_x + n_y + m_\ell) + m^2. \quad (6)$$

In the NC complex space the Klein-Gordon oscillator is described by the following equation:

$$\begin{pmatrix} (2p_z + im\omega \hat{\bar{z}})(2p_{\bar{z}} - im\omega \hat{z}) & 0 \\ 0 & (2p_{\bar{z}} - im\omega \hat{z})(2p_z + im\omega \hat{\bar{z}}) \end{pmatrix} \psi = (E^2 - m^2)\psi. \quad (7)$$

To solve this equation we use the NC complex coordinates:

$$\hat{z} = z + i\theta p_{\bar{z}}, \quad (8)$$

$$\hat{\bar{z}} = \bar{z} - i\theta p_z, \quad (9)$$

$$\hat{p}_z = p_z, \quad \hat{p}_{\bar{z}} = p_{\bar{z}}. \quad (10)$$

Inserting eqs. (8)-(10) into eq. (7), we have:

$$\left[\left(1 + \frac{m\omega\theta}{2} \right)^2 (p_x^2 + p_y^2) + m^2\omega^2 (x^2 + y^2) - 2m\omega L_z - m^2\omega^2\theta (L_z \pm 1) \right] \psi = (E^2 - m^2 + 2m\omega) \psi. \quad (11)$$

The energy eigenvalues are given by:

$$E^2 = 2m\omega_\theta (n_x + n_y + 1) + 2m\omega_\theta (m_\ell \pm 1) + m^2, \quad (12)$$

with $\omega_\theta = \omega(1 - m\omega\theta/2)$. Such effects are similar to the normal Zeeman splitting of a particle with spin 1/2 and thus the degeneracy of energy levels is completely removed. The oscillator is positioned in the four equivalent points $(z \uparrow, \bar{z} \uparrow, z \downarrow, \bar{z} \downarrow) \Leftrightarrow (z, \bar{z}, -z, -\bar{z})$. Therefore the eigenfunction $\psi(z, \bar{z})$ takes values in C^4 , spin up, spin down, particle, antiparticle. This oscillator is described by two double-component spinors [4, 6]:

$$\psi_{n0} \begin{pmatrix} \psi_{n0}^+ \\ \psi_{n0}^- \end{pmatrix}, \quad \text{and} \quad \psi_{0n} \begin{pmatrix} \psi_{0n}^+ \\ \psi_{0n}^- \end{pmatrix},$$

where the sign (\pm) signifies spin up or down, and the wave functions ψ_{n0} and ψ_{0n} have the following form:

$$\psi_{n0}(z, \bar{z}) = \sqrt{\frac{(m\omega)^{n+1}}{\pi n!}} z^n \exp\left(-\frac{m\omega}{2} z\bar{z}\right), \quad (13a)$$

$$\psi_{0n}(z, \bar{z}) = \sqrt{\frac{(m\omega)^{n+1}}{\pi n!}} \bar{z}^n \exp\left(-\frac{m\omega}{2} z\bar{z}\right). \quad (13b)$$

3 2D Klein-Gordon oscillator in NC complex phase space

In a NC complex phase space we replace the coordinate and momentum operators (relations (8-10)) by:

$$\hat{z} = z + i\theta p_{\bar{z}}, \quad (14)$$

$$\hat{\bar{z}} = \bar{z} - i\theta p_z, \quad (15)$$

$$\hat{p}_z = p_z + i\frac{\bar{\theta}}{4} \bar{z}, \quad (16)$$

$$\hat{p}_{\bar{z}} = p_{\bar{z}} - i\frac{\bar{\theta}}{4} z. \quad (17)$$

Inserting eqs. (14)-(17) into eq. (7), we have two equations:

$$\left[\left(1 - \frac{m\omega\theta}{2} \right)^2 (p_x^2 + p_y^2) + m^2\omega^2 \left(1 + \frac{\bar{\theta}}{2m\omega} \right)^2 (x^2 + y^2) + 2m\omega L_z - m^2\omega^2 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2} \right) (L_z + 1) \right] \psi = (E^2 - m^2 + 2m\omega) \psi, \quad (18)$$

and

$$\left[\left(1 - \frac{m\omega\theta}{2} \right)^2 (p_x^2 + p_y^2) + m^2\omega^2 \left(1 + \frac{\bar{\theta}}{2m\omega} \right)^2 (x^2 + y^2) + 2m\omega L_z - m^2\omega^2 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2} \right) (L_z - 1) \right] \psi = (E^2 - m^2 + 2m\omega) \psi. \quad (19)$$

The equations (18) and (19) are similar to the equation of motion for a fermion of spin 1/2 in a constant magnetic field. Under these conditions the equations (18) and (19) take the following form:

$$\left[\left(1 + \frac{m\omega\theta}{2} \right)^2 (p_x^2 + p_y^2) + m^2\omega^2 \left(1 + \frac{\bar{\theta}}{2m\omega} \right)^2 (x^2 + y^2) - 2m\omega L_z - m^2\omega^2 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2} \right) (L_z + 2s_z) \right] \psi = (E^2 - m^2 + 2m\omega) \psi, \quad (20)$$

with $s_z = \pm 1/2$. The energy eigenvalues are given by:

$$E^2 = 2m\Omega_\theta (n_x + n_y + 1) - 2m\omega (m_\ell + 1) - m^2\omega^2 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2} \right) (m_\ell \pm 1) + m^2, \quad (21)$$

where

$$\Omega_\theta = \omega \left(1 + \frac{m\omega\theta}{2} \right) \left(1 + \frac{\bar{\theta}}{2m\omega} \right). \quad (22)$$

We have thus shown that the non-commutativity effects are manifested in energy levels and thus the degeneracy of the levels is completely removed, so that they are split into $(2m_\ell)$ levels, similarly to the effects of a magnetic field interacting automatically with the spin of a particle.

4 Thermodynamic properties of the 2D Klein-Gordon oscillator in NC complex phase space

The thermodynamic functions associated with the NC complex oscillator are also of interest. The eigenvalues of the 2D Klein-Gordon oscillator in NC complex phase space are [7]:

$$E^\pm = \pm m\sqrt{\lambda_{\theta\bar{\theta}} + \gamma_{\theta\bar{\theta}}n}, \quad n = 0, 1, 2, \dots, \quad (23)$$

where

$$\lambda_{\theta\bar{\theta}}^\ell = 1 + \gamma_{\theta\bar{\theta}} - \frac{2\omega}{m}(\ell + 1) - \omega^2 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2} \right) (\ell \pm 1), \quad \ell = 0, 1, \dots, \quad (24)$$

and

$$\gamma_{\theta\bar{\theta}} = 2 \frac{\Omega_\theta}{m}. \quad (25)$$

We concentrate, firstly, on the calculation of the partition function $Z(\beta, \theta, \bar{\theta})$, defined as:

$$Z(\beta, \theta, \bar{\theta}) = \sum_{n,s} \exp[-\beta(E_{n,s} - E_{0,s})], \quad (26)$$

where $\beta = 1/k_B T$ is the Boltzmann factor, and E_0 is the background energy correspond to $n = 0$. Therefore we have the single-oscillator partition function with $\ell = 0$, from eq. (26):

$$\begin{aligned} Z(\beta, \theta, \bar{\theta}) &= \sum_{n=0}^{\infty} \exp \left[-\beta m \left(\sqrt{1 + \gamma_{\theta\bar{\theta}} n} - 1 \right) \right] + \\ &+ \sum_{n=0}^{\infty} \exp \left[-\beta m \left(\sqrt{\lambda_{\theta\bar{\theta}} + \gamma_{\theta\bar{\theta}} n} - \sqrt{\lambda_{\theta\bar{\theta}}} \right) \right], \end{aligned} \quad (27)$$

where

$$\lambda_{\theta\bar{\theta}} = 1 + 2\omega^2 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2} \right). \quad (28)$$

On the other hand, the Euler-Maclaurin formula [8] is:

$$\sum_{x=0}^{\infty} f(x) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)dx - \sum_{p=1}^{\infty} \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0), \quad (29)$$

where

$$B_{2n} = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{p=1}^{\infty} p^{-2n}, \quad (30)$$

are the Bernoulli numbers. Using the Euler-Maclaurin formula (eq. (29)), and after a simple calculation, the partition function in eq. (27) can be written as:

$$\begin{aligned} Z(\beta, \theta, \bar{\theta}) &= 1 + \frac{2}{\gamma_{\theta\bar{\theta}}\beta^2} \left[(1 + \beta) + \left(1 + \beta\sqrt{\lambda_{\theta\bar{\theta}}} \right) \right] + \\ &- \frac{B_2}{2} \left(e^\beta f_1^{(1)} + e^{\beta\sqrt{\lambda_{\theta\bar{\theta}}}} f_2^{(1)} \right) + \\ &- \frac{B_4}{24} \left(e^\beta f_1^{(3)} + e^{\beta\sqrt{\lambda_{\theta\bar{\theta}}}} f_2^{(3)} \right) + \dots, \end{aligned} \quad (31)$$

where

$$f_1^{(1)} = -\frac{\gamma_{\theta\bar{\theta}}\beta m}{2} e^{-\beta}, \quad (32a)$$

$$f_2^{(1)} = -\frac{\gamma_{\theta\bar{\theta}}\beta m}{2\sqrt{\lambda_{\theta\bar{\theta}}}} e^{-\beta\sqrt{\lambda_{\theta\bar{\theta}}}}, \quad (32b)$$

and

$$f_1^{(3)} = \left[\frac{-3\beta m (\gamma_{\theta\bar{\theta}})^3}{8} - \frac{3\beta^2 m^2 (\gamma_{\theta\bar{\theta}})^3}{8} - \frac{3\beta^3 m^3 (\gamma_{\theta\bar{\theta}})^3}{8} \right] e^{-\beta}, \quad (33a)$$

$$f_2^{(3)} = \left[\frac{-3\beta m (\gamma_{\theta\bar{\theta}})^3}{8 (\lambda_{\theta\bar{\theta}})^{5/2}} - \frac{3\beta^2 m^2 (\gamma_{\theta\bar{\theta}})^3}{8 (\lambda_{\theta\bar{\theta}})^2} - \frac{3\beta^3 m^3 (\gamma_{\theta\bar{\theta}})^3}{8 (\lambda_{\theta\bar{\theta}})^{3/2}} \right] e^{-\beta\sqrt{\lambda_{\theta\bar{\theta}}}}, \quad (33b)$$

with $B_2 = 1/6$ and $B_4 = -1/30$. By replacing Eqs. (32)–(33) into Eq. (31), we obtain the partition function in NC complex phase space as:

$$\begin{aligned} Z(\beta, \theta, \bar{\theta}) = & 1 + \frac{2}{\gamma_{\theta\bar{\theta}}} \left(1 + \sqrt{\lambda_{\theta\bar{\theta}}} \right) \beta^{-1} + \frac{4}{\gamma_{\theta\bar{\theta}}} \beta^{-2} + \\ & + \frac{\gamma_{\theta\bar{\theta}}}{24} \left[\left(1 + \frac{1}{\sqrt{\lambda_{\theta\bar{\theta}}}} \right) - \frac{(\gamma_{\theta\bar{\theta}})^2}{80} \left(1 + \frac{1}{(\lambda_{\theta\bar{\theta}})^{5/2}} \right) \right] \beta + \\ & - \frac{(\gamma_{\theta\bar{\theta}})^3}{1920} \left(1 + \frac{1}{(\lambda_{\theta\bar{\theta}})^2} \right) \beta^2 - \frac{(\gamma_{\theta\bar{\theta}})^3}{1920} \left(1 + \frac{1}{(\lambda_{\theta\bar{\theta}})^{3/2}} \right) \beta^3. \end{aligned} \quad (34)$$

Hence there is a characteristic temperature which divides the temperature range into two regions: $\beta \gg \beta_0 = 1/mc^2$ for very low temperatures, and $\beta \ll \beta_0$ for very high temperatures. In this context, we derive the thermodynamic properties of our system, such as the total energy, entropy, free energy and specific heat, which are given by:

$$\langle E(\beta, \theta, \bar{\theta}) \rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \bar{\theta}), \quad C_V = -k_B \beta^2 \frac{\partial \langle E(\beta, \theta, \bar{\theta}) \rangle}{\partial \beta}, \quad (35a)$$

$$F = -\frac{1}{\beta} \ln Z(\beta, \theta, \bar{\theta}), \quad S = -\frac{1}{T} F - k_B \frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \bar{\theta}). \quad (35b)$$

4.1 Results and discussions

For very low temperatures the partition function in eq. (34) can be written as:

$$\begin{aligned} Z(\beta, \theta, \bar{\theta}) \simeq & \frac{\gamma_{\theta\bar{\theta}}}{24} \left[\left(1 + \frac{1}{\sqrt{\lambda_{\theta\bar{\theta}}}} \right) - \frac{(\gamma_{\theta\bar{\theta}})^2}{80} \left(1 + \frac{1}{(\lambda_{\theta\bar{\theta}})^{5/2}} \right) \right] \beta + \\ & - \frac{(\gamma_{\theta\bar{\theta}})^3}{1920} \left(1 + \frac{1}{(\lambda_{\theta\bar{\theta}})^2} \right) \beta^2 - \frac{(\gamma_{\theta\bar{\theta}})^3}{1920} \left(1 + \frac{1}{(\lambda_{\theta\bar{\theta}})^{3/2}} \right) \beta^3. \end{aligned} \quad (36)$$

The mean energy $\langle E(\beta, \theta, \bar{\theta}) \rangle$ of the systems is

$$\langle E(\beta, \theta, \bar{\theta}) \rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \bar{\theta}) \sim 0, \quad (37)$$

and the specific heat C_V of the systems is:

$$C_V = -k_B \beta^2 \frac{\partial \langle E(\beta, \theta, \bar{\theta}) \rangle}{\partial \beta} \sim -3k_B, \quad (38)$$

which is clearly a negative quantity. In other words this means that the Klein-Gordon oscillator in NC complex phase space is similar to a self-gravitating system that is discussed in the framework of a non-extensive kinetic theory (see [9] and references therein).

The free energy F is given by:

$$F = -\frac{1}{\beta} \ln Z(\beta, \theta, \bar{\theta}) \sim 0, \quad (39)$$

and the entropy of the systems is:

$$S = -\frac{1}{T} F - k_B \frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \bar{\theta}) \sim 0. \quad (40)$$

For very high temperatures the partition function in eq. (34) can be written as:

$$Z(\beta, \theta, \bar{\theta}) \simeq 1 + \frac{2}{\gamma_{\theta\bar{\theta}}} \left(1 + \sqrt{\lambda_{\theta\bar{\theta}}}\right) \beta^{-1} + \frac{4}{\gamma_{\theta\bar{\theta}}} \beta^{-2}, \quad (41)$$

and the mean energy $\langle E(\beta, \theta, \bar{\theta}) \rangle$, and the specific heat C_V of the systems are given by:

$$\langle E(\beta, \theta, \bar{\theta}) \rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \bar{\theta}) \sim 2\beta^{-1}, \quad (42)$$

$$C_V = \frac{\partial \langle E(\beta, \theta, \bar{\theta}) \rangle}{\partial T} \sim 2k_B. \quad (43)$$

Note that these results are similar to those of the Dirac oscillator under a magnetic field in a NC space [5].

We shown in figures 1, 2 and 3 comparisons of the partition function Z as a function of β , the thermodynamic function E/mc^2 of the Klein Gordon Bosons as a function of τ and the heat capacity C_V/k_B of Klein-Gordon Bosons as a function of τ , for different values of θ and $\bar{\theta}$.

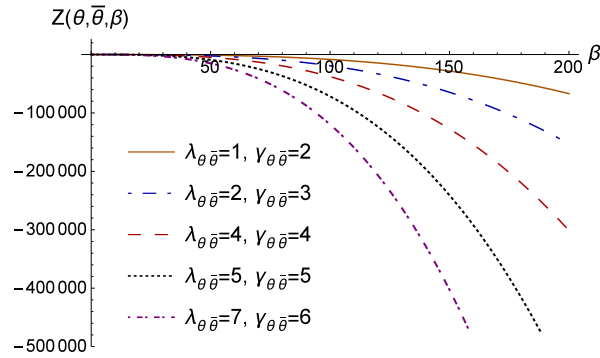


Figure 1: Comparison of the partition function Z as a function of β for different values of θ and $\bar{\theta}$.

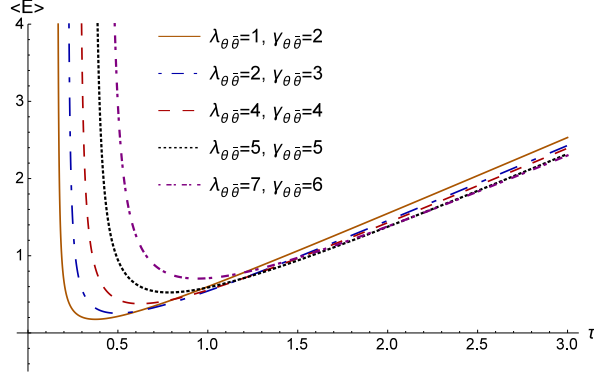


Figure 2: Comparison of the thermodynamic function E/mc^2 of the Klein Gordon Bosons as a function of τ for different values of θ and $\bar{\theta}$.

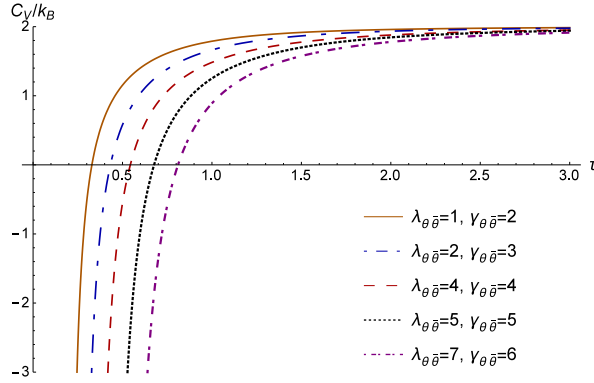


Figure 3: Comparison of the heat capacity C_V/k_B of Klein-Gordon Bosons as a function of τ for different values of θ and $\bar{\theta}$.

5 Conclusions

In this paper we started from a Klein-Gordon oscillator in a NC complex phase space. Using the Moyal product method, we derived the deformed Klein-Gordon oscillator and showed that it is similar to the Klein-Gordon equation for a particle with spin 1/2 in a uniform magnetic field in NC phase space [2]. We solved this equation exactly and found that the NC energy levels split into $2m_\ell$ levels. Thus the system without spin in a NC complex coordinate space has an added advantage that the spin effect is automatically manifested. The statistical quantities of the 2D Klein-Gordon oscillator in a NC complex phase space were investigated and the effect of the NC parameter on thermal properties was discussed. It was found that the dependence of the specific heat C_V on the NC parameter gives rise to a negative quantity. Phenomenologically, this effectively confirms the presence of the effects of self-gravitation at this level.

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